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Configurational equilibrium of circular-arc cracks with surface stress

Chien H. Wu *, Ming L. Wang

Department of Civil and Materials Engineering (MC 246), University of Illinois at Chicago, P.O. Box 4348, 842 West Taylor Street, Chicago, IL 60607-7023, USA

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Abstract

Solid surface stresses are known to behave like the prestress in a prestressed membrane that is perfectly fitted on the bounding surface of a bulk material. The inclusion of such a surface stress in an otherwise traction-free crack surface leads to additional loads for the bulk material: a pair of point forces, one at each crack tip, a uniformly distributed compressive load on the convex side of the crack, and a uniformly distributed tensile load on the concave side. As a result, the values of the stress intensity factors are altered, and the crack-tip stress fields become r^{-1} singular, in addition to being $r^{-1/2}$ singular. The severity of the added singularity does not carry any particular physical significance in that the configurational equilibrium is always an energy condition and never a stress criterion. Indeed, the new configurational equilibrium condition – or fracture criterion – is also dependent on the surface stress as well as the curvature of the crack. The dependence on curvature becomes more and more pronounced as the radius of curvature becomes smaller and smaller.

Keywords: Configurational equilibrium; Surface stress; Circular-arc crack

1. Introduction

When an infinite solid, characterized by a shear modulus μ and a Poisson's ratio v, with a circular-arc crack of radius ρ and central angle 2ϕ is loaded by remote stresses, the total potential energy Π may be written as

$$\Pi = \Pi_0 + \Delta\Pi, \tag{1.1}$$

where Π_0 is the potential of the solid without the crack. Similarly, the total elastic strain energy U_e is

$$U_e = U_{e0} + \Delta U_e, \tag{1.2}$$

where U_{e0} is the strain energy of the solid without the crack. Since the solid is load controlled by the remote stresses, we have

* Corresponding author. Tel.: +1-312-413-2644; fax: +1-312-996-2426.
E-mail address: cwu@uic.edu (C.H. Wu).

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$$\Delta\Pi = -\Delta U_e + 4\Gamma_0\rho\phi, \tag{1.3}$$

where Γ_0 is the surface energy density per unit area, or surface tension. For configurational equilibrium of a circularly extending crack, we have

$$\frac{1}{2\rho}\frac{\partial\Delta\Pi}{\partial\phi} = -G + 2\Gamma_0 = 0, \tag{1.4}$$

where

$$G = \frac{1}{2\rho}\frac{\partial\Delta U_e}{\partial\phi} \tag{1.5}$$

is the (strain) energy release rate. Moreover, for plane-elasticity problems,

$$G = \frac{\kappa+1}{8\mu}\left(K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2\right), \tag{1.6}$$

where

$$\kappa = \begin{cases} 3-4\nu & \text{for plane strain} \\ (3-\nu)/(1+\nu) & \text{for plane stress,} \end{cases} \tag{1.7}$$

and K_{I} and K_{II} are the stress intensity factors (SIF). Condition (1.4) may be referred to as a K-criterion in that G consists of the sum of the squares of the SIFs and

$$K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2 = K_{\mathrm{IC}}^2 \equiv \frac{16\mu\Gamma_0}{\kappa+1}. \tag{1.8}$$

The purpose of this paper is to show that in the presence of surface stress, neither Eq. (1.6) nor Eq. (1.8) is a valid identity.

In general, the surface energy density is a function of the surface deformation (Shuttleworth, 1950; Herring, 1951; Wu, 1996a,b; Wu et al., 1998). For a linear two-dimensional theory, the surface energy density Γ depends on a single surface strain ε. Since the surface stress, which is defined as the derivative of Γ with respect to ε, is known to be of a residual-stress type, the minimum requirement for Γ is a linear function of ε, viz.

$$\Gamma(\varepsilon) = \Gamma_0 + \Sigma_0\varepsilon, \tag{1.9}$$

where Σ_0 is the surface stress coefficient or surface stress for short. The presence of Σ_0 on a straight crack leads to a pair of crack-tip point loads of magnitude $2\Sigma_0$. It was shown by Wu (1999) that, with the inclusion of the $2\Sigma_0$ forces, the new potential for a straight crack under a mode-I loading implied by $K_{\mathrm{I}} = \sigma_{22}\sqrt{\pi a}$ is

$$\Delta\Pi = -\frac{\kappa+1}{8\mu}\left(\pi a^2\sigma_{22}^2 + 4\Sigma_0\sigma_{22}a\right) + 4a\Gamma_0. \tag{1.10}$$

The associated configurational equilibrium condition becomes

$$K_{\mathrm{I}}^2 + 2\frac{\Sigma_0}{\sqrt{\pi a}}K_{\mathrm{I}} - K_{\mathrm{IC}}^2 = 0 \Rightarrow \sigma_{22}\sqrt{\pi a} \approx K_{\mathrm{IC}} - \frac{\Sigma_0}{\sqrt{\pi a}} + \cdots \tag{1.11}$$

It is noted that while the crack-tip point loads do not alter the value of K_{I} in any manner, they actually appear in the configurational equilibrium condition. The situation for curvilinear cracks is even more intricate in that the crack surfaces are now subjected to the normal traction induced by the surface stress. As a result, the values of K_{I} and K_{II} also depend on Σ_0. The circular-arc counterpart of Eq. (1.10) is analyzed in detail in this paper.

Elastic fields induced by surface stresses have been considered by Andreussi and Gurtin (1977), Dunham and Gurtin (1977), Murdoch (1976, 1977), and Gurtin and Murdoch (1978). Chuang (1987) first estimated the nonlinear effects of surface stress on stress intensity factors. Our interest has been in examining the effect of surface stress on the equilibrium and evolution of configurations. The calculation of generalized configurational forces has therefore been our primary concern. This involves the nonlinear surface chemical potential (Wu, 1996a,b), the stability of planar surfaces (Wu et al., 1998), and the configurational equilibrium of voids and cracks (Wu, 1999). In particular, the effect of crack-tip point loads on fracture was examined in a recent study by Wu and Wang (2000).

2. A circular-arc crack with surface stress

2.1. Formulation

Let x_i $(i = 1, 2, 3)$ be rectangular Cartesian coordinates and $\mathbf{e_i}$ the associated unit vectors. The associated cylindrical coordinates (r, θ, x_3) are defined by $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$. We consider plane elasticity problems so that the displacements u_α, strains $\varepsilon_{\alpha\beta}$ and stresses $\tau_{\alpha\beta}$ are functions of x_β where Greek subscripts range from 1 to 2. The deformation of a circular region of radius R containing a circular-arc crack of central angle 2ϕ is considered (Fig. 1a). The crack C is on the circle $r = \rho$ so that

$$C: \ r = \rho, \quad -\phi \leqslant \theta \leqslant +\phi. \tag{2.1}$$

It is implicitly assumed that $\rho/R \ll 1$ and tends to zero when the circle is actually infinite.

In terms of the cylindrical coordinates and components the traction condition on $r = R$ is

$$\tau_{rr}(R, \theta) = \sigma_{rr}(\theta) \equiv \tfrac{1}{2}[(\sigma_{22} + \sigma_{11}) - (\sigma_{22} - \sigma_{11})\cos 2\theta], \tag{2.2}$$

$$\tau_{r\theta}(R, \theta) = \sigma_{r\theta}(\theta) \equiv \tfrac{1}{2}(\sigma_{22} - \sigma_{11})\sin 2\theta, \tag{2.3}$$

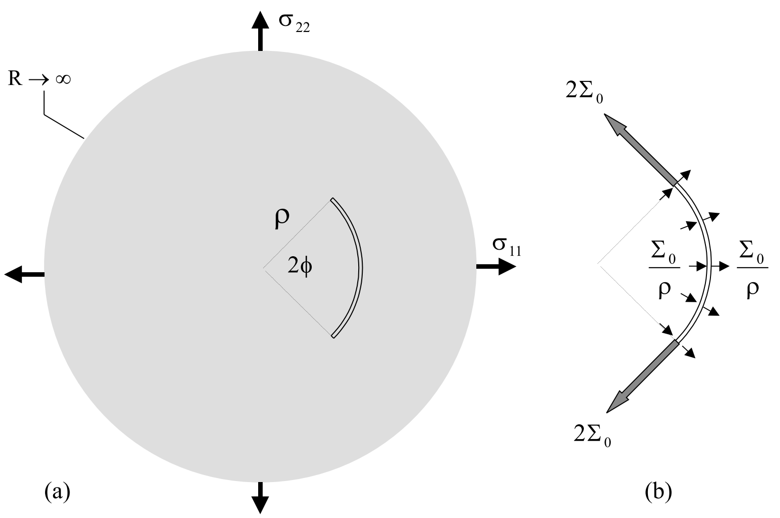

Fig. 1. (a) A circular-arc crack in a large circular specimen subjected to remote loading, and (b) the free body of the circular-arc 'surface'. The load on the bulk is the reverse of that shown in (b).

where σ_{11} and σ_{22} are the given constants. The above conditions are setup for the convenience that, as R becomes infinite, the constant stresses at infinity are

$$\tau_{11} = \sigma_{11}, \quad \tau_{22} = \sigma_{22}, \quad \tau_{12} = 0. \tag{2.4}$$

The crack faces are assumed to have the surface energy density $\Gamma(\varepsilon)$ defined by Eq. (1.9), where the surface strain ε is tied to the bulk strain by

$$\varepsilon = \varepsilon^{\pm}(\theta) \equiv \varepsilon_{\theta\theta}(\rho \pm 0, \theta) \quad \text{for } -\phi \leqslant \theta \leqslant +\phi. \tag{2.5}$$

The surfaces are therefore under the action of a prestress of magnitude Σ_0. It is termed a stress, but its dimension is only force per length. While there is no uniformity in the nomenclature for the quantities Γ_0 and Σ_0, we refer to them as the surface tension and surface stress, respectively. For liquids Σ_0 is zero and the surface energy is just the surface tension, a commonly accepted terminology. Since the crack is circular, the prestressed crack surface, considered as a free body, is under the action of two end forces, each of magnitude $2\Sigma_0$, and a distributed surface load of magnitude Σ_0/ρ on each of the two sheets of circular surfaces (Fig. 1b). Thus for the bulk material (without the *surface*), the surface traction is just the opposite of what is indicated on the free body of the *surface* (Fig. 1b).

2.2. Elastic deformation of the bulk

Let $(u_\alpha, \varepsilon_{\alpha\beta}, \tau_{\alpha\beta})$ denote the desired solution of the bulk elastic medium (without the surface). The solution satisfies the boundary conditions

$$\tau_{rr}(R,\theta) = \sigma_{rr}(\theta), \qquad \tau_{r\theta}(R,\theta) = \sigma_{r\theta}(\theta), \tag{2.6}$$

$$\tau_{rr}(\rho \pm 0, \theta) = \pm\Sigma_0/\rho, \quad \tau_{r\theta}(\rho \pm 0, \theta) = 0 \quad \text{for } -\phi \leqslant \theta \leqslant +\phi, \tag{2.7}$$

$$\text{tensile point load } 2\Sigma_0 \text{ at } (\rho, \pm\phi), \tag{2.8}$$

where $\sigma_{rr}(\theta)$ and $\sigma_{r\theta}(\theta)$ are defined by Eqs. (2.2) and (2.3). For the purpose of convenience, the above elastic field can be split into two separate fields as follows:

$$\left(u_\alpha, \varepsilon_{\alpha\beta}, \tau_{\alpha\beta}\right) = \left(u_\alpha^{(o)}, \varepsilon_{\alpha\beta}^{(o)}, \tau_{\alpha\beta}^{(o)}\right) + \left(u_\alpha^{(s)}, \varepsilon_{\alpha\beta}^{(s)}, \tau_{\alpha\beta}^{(s)}\right), \tag{2.9}$$

where

$$\tau_{rr}^{(o)}(R,\theta) = \sigma_{rr}(\theta), \qquad \tau_{r\theta}^{(o)}(R,\theta) = \sigma_{r\theta}(\theta), \tag{2.10}$$

$$\tau_{rr}^{(o)}(\rho \pm 0, \theta) = \tau_{r\theta}^{(o)}(\rho \pm 0, \theta) = 0 \quad \text{for } -\phi \leqslant \theta \leqslant +\phi, \tag{2.11}$$

and

$$\tau_{rr}^{(s)}(R,\theta) = 0, \qquad \tau_{r\theta}^{(s)}(R,\theta) = 0, \tag{2.12}$$

$$\tau_{rr}^{(s)}(\rho \pm 0, \theta) = \pm\Sigma_0/\rho, \qquad \tau_{r\theta}^{(s)}(\rho \pm 0, \theta) = 0 \quad \text{for } -\phi \leqslant \theta \leqslant +\phi, \tag{2.13}$$

$$\text{tensile point load } 2\Sigma_0 \text{ at } (\rho, \pm\phi). \tag{2.14}$$

It is clear that Eqs. (2.10) and (2.11) are the boundary conditions of an ordinary crack problem in the sense that Griffith solution (modified for circular-arc cracks) is recovered as R tends to infinity. The conditions (2.12)–(2.14) represent the effect of surface stress. We proceed to calculate the energies of the three elastic fields defined by Eq. (2.9).

For plane elasticity, the stress–strain relations and the strain energy density W may be conveniently written in terms of μ and κ of Eq. (1.7). They are

$$\tau_{\alpha\beta} = 2\mu\left[\varepsilon_{\alpha\beta} + \frac{3-\kappa}{2(\kappa-1)}\delta_{\alpha\beta}\varepsilon_{\gamma\gamma}\right], \tag{2.15}$$

$$W\left(\varepsilon_{\alpha\beta}\right) = \mu\left[\frac{3-\kappa}{2(\kappa-1)}\varepsilon_{\alpha\alpha}\varepsilon_{\beta\beta} + \varepsilon_{\alpha\beta}\varepsilon_{\alpha\beta}\right], \tag{2.16}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. The elastic energies are defined as follows:

$$\begin{aligned} U_{\mathrm{e}} &\equiv \int W(\varepsilon_{\alpha\beta})\,\mathrm{d}A \\ &= +\frac{R}{2}\int_0^{2\pi}[\sigma_{rr}(\theta)u_r(R,\theta) + \sigma_{r\theta}(\theta)u_\theta(R,\theta)]\,\mathrm{d}\theta - \frac{1}{2}(2\Sigma_0)[u_\theta(\rho,\phi) - u_\theta(\rho,-\phi)] \\ &\quad - \frac{1}{2}\int_{-\phi}^{+\phi}\frac{\Sigma_0}{\rho}[u_r(\rho+0,\theta) + u_r(\rho-0,\theta)]\rho\,\mathrm{d}\theta, \end{aligned} \tag{2.17}$$

$$U_{\mathrm{e}}^{(\mathrm{o})} \equiv \int W\left(\varepsilon_{\alpha\beta}^{(\mathrm{o})}\right)\mathrm{d}A = \frac{R}{2}\int_0^{2\pi}\left[\sigma_{rr}(\theta)u_r^{(\mathrm{o})}(R,\theta) + \sigma_{r\theta}(\theta)u_\theta^{(\mathrm{o})}(R,\theta)\right]\mathrm{d}\theta, \tag{2.18}$$

$$\begin{aligned} U_{\mathrm{e}}^{(\mathrm{s})} &\equiv \int W\left(\varepsilon_{\alpha\beta}^{(\mathrm{s})}\right)\mathrm{d}A \\ &= -\frac{1}{2}(2\Sigma_0)\left[u_\theta^{(\mathrm{s})}(\rho,\phi) - u_\theta^{(\mathrm{s})}(\rho,-\phi)\right] - \frac{1}{2}\int_{-\phi}^{+\phi}\frac{\Sigma_0}{\rho}\left[u_r^{(\mathrm{s})}(\rho+0,\theta) + u_r^{(\mathrm{s})}(\rho-0,\theta)\right]\rho\,\mathrm{d}\theta. \end{aligned} \tag{2.19}$$

Since the displacement at the point of application of a point load is logarithmically singular, the $u_\theta(\rho,\pm\phi)$ terms are actually infinite. Also, as R tends to infinity, Eq. (2.18) and the first term of Eq. (2.17) tend to infinity like R^2. These infinite energies, however, do not contribute to the configurational equilibrium of the crack. Substituting Eq. (2.9) into Eq. (2.17) and applying Eqs. (2.18) and (2.19), we get

$$U_{\mathrm{e}} = U_{\mathrm{e}}^{(\mathrm{o})} + U_{\mathrm{e}}^{(\mathrm{s})} + U_{\mathrm{e}}^{(\mathrm{os})}, \tag{2.20}$$

where $U_{\mathrm{e}}^{(\mathrm{os})}$ may be called the elastic interaction energy between the o- and the s-system. It has the form

$$\begin{aligned} U_{\mathrm{e}}^{(\mathrm{os})} &= -\Sigma_0[u_\theta(\rho,\phi) - u_\theta(\rho,-\phi)] - \frac{\Sigma_0}{2}\int_{-\phi}^{+\phi}\left[u_r^{(\mathrm{o})}(\rho+0,\theta) + u_r^{(\mathrm{o})}(\rho-0,\theta)\right]\mathrm{d}\theta + \frac{R}{2} \\ &\quad \times \int_0^{2\pi}\left[\sigma_{rr}(\theta)u_r^{(\mathrm{s})}(R,\theta) + \sigma_{r\theta}(\theta)u_\theta^{(\mathrm{s})}(R,\theta)\right]\mathrm{d}\theta \end{aligned} \tag{2.21}$$

or

$$U_{\mathrm{e}}^{(\mathrm{os})} = -2\Sigma_0[u_\theta(\rho,\phi) - u_\theta(\rho,-\phi)] - \Sigma_0\int_{-\phi}^{+\phi}\left[u_r^{(\mathrm{o})}(\rho+0,\theta) + u_r^{(\mathrm{o})}(\rho-0,\theta)\right]\mathrm{d}\theta, \tag{2.22}$$

where the last equality follows from the reciprocal theorem of Betti and Rayleigh. The second integral of Eq. (2.21) is not convenient to use because $u_r^{(\mathrm{s})}$ and $u_\theta^{(\mathrm{s})}$ must be solved for finite R to the accuracy of $(\rho/R)^2$ (Sih and Liebowitz, 1967).

2.3. Potential energy of the bulk-surface system

The potential energy Π for the system of Fig. 1a is

$$\Pi = U_e + U_s - V, \tag{2.23}$$

where

$$U_e = \int W(\varepsilon_{\alpha\beta})\,\mathrm{d}A, \qquad U_s = 2\rho\int_{-\phi}^{+\phi}\Gamma(\varepsilon)\,\mathrm{d}\theta, \tag{2.24}$$

and

$$V = R\int_0^{2\pi}[\sigma_{rr}(\theta)u_r(R,\theta) + \sigma_{r\theta}(\theta)u_\theta(R,\theta)]\,\mathrm{d}\theta \tag{2.25}$$

is the potential associated with the remote loading. Using Eqs. (1.9) and (2.5), we obtain from the second part of Eq. (2.24) the total surface energy

$$U_s = 4\rho\phi\Gamma_0 + 2\Sigma_0[u_\theta(\rho,\phi) - u_\theta(\rho,-\phi)] + \Sigma_0\int_{-\phi}^{+\phi}[u_r(\rho+0,\theta) + u_r(\rho-0,\theta)]\,\mathrm{d}\theta. \tag{2.26}$$

Substituting Eqs. (2.20), (2.25), (2.26) into Eq. (2.23) and applying Eqs. (2.17)–(2.19), we finally obtain

$$\Pi = -U_e^{(o)} - U_e^{(s)} - U_e^{(os)} + 4\rho\phi\Gamma_0. \tag{2.27}$$

According to Eq. (2.20), the above equation is simply $\Pi = -U_e + 4\rho\phi\Gamma_0$. It should be noted, however, that neither $4\rho\phi\Gamma_0$ is the total surface energy, nor $-U_e$ the total potential associated with the total elastic field (cf. Eqs. (2.25) and (2.26)). It is now necessary to find the two solutions governed by Eqs. (2.10)–(2.14) for $R = \infty$.

3. Circular-arc crack under biaxial tension

3.1. Complex formulation

In terms of the complex variable $z = x_1 + \mathrm{i}x_2$, the following relations hold (England, 1971; Wu, 1994):

$$2\mu(u_r + \mathrm{i}u_\theta)\mathrm{e}^{\mathrm{i}\theta} = \kappa W(z) - W\left(\frac{\rho^2}{\bar{z}}\right) + \left(\frac{\rho^2}{\bar{z}} - z\right)\overline{W'(z)} + f\left(\frac{\rho^2}{\bar{z}}\right), \tag{3.1}$$

$$\tau_{rr} + \mathrm{i}\tau_{r\theta} = W'(z) + \frac{\rho^2}{z\bar{z}}W'\left(\frac{\rho^2}{\bar{z}}\right) + \left(1 - \frac{\rho^2}{z\bar{z}}\right)\left[\overline{W'(z)} - \bar{z}\overline{W''(z)}\right] - \frac{\rho^2}{z\bar{z}}f'\left(\frac{\rho^2}{\bar{z}}\right), \tag{3.2}$$

where W and f are holomorphic functions on the z-plane with cuts along the circle $r = \rho$. The function f is actually the result of an analytic continuation defined in terms of W and another holomorphic function w, viz.

$$f(z) = W(z) - z\overline{W'\left(\frac{\rho^2}{\bar{z}}\right)} - \overline{w\left(\frac{\rho^2}{\bar{z}}\right)}. \tag{3.3}$$

Some of the solution properties are more conveniently expressed in terms of W and w, instead of f. For example, the properties for large $|z|$ are given by

$$W(z) = W_1 z + W_{-1}\frac{1}{z} + \cdots, \tag{3.4}$$

$$w(z) = w_1 z + w_{-1}\frac{1}{z} + \cdots, \tag{3.5}$$

where

$$W_1 = A \equiv \tfrac{1}{4}(\sigma_{22} + \sigma_{11}), \tag{3.6}$$

$$w_1 = C + \mathrm{i}D \equiv \tfrac{1}{2}(\sigma_{22} - \sigma_{11}) + \mathrm{i}\sigma_{12}, \tag{3.7}$$

and $\sigma_{\alpha\beta}$ are the constant stresses at infinity.

For the crack defined by Eq. (2.1), the traction boundary conditions, expressed in terms of W and f, are

$$f'^{+} - f'^{-} = \left(\tau_{rr}^{+} - \tau_{rr}^{-}\right) + \mathrm{i}\left(\tau_{r\theta}^{+} - \tau_{r\theta}^{-}\right), \tag{3.8}$$

$$\left(2W' - f'\right)^{+} + \left(2W' - f'\right)^{-} = \left(\tau_{rr}^{+} + \tau_{rr}^{-}\right) + \mathrm{i}\left(\tau_{r\theta}^{+} + \tau_{r\theta}^{-}\right), \tag{3.9}$$

where $F^{\pm}(\rho,\theta) \equiv F(\rho \pm 0, \theta)$ for $(-\phi \leqslant \theta \leqslant +\phi)$. The function $f'(z)$ has a pole of second order at the origin,

$$f'(z) = C\frac{\rho^2}{z^2} + \cdots \quad \text{as } z \to 0, \tag{3.10}$$

and tends to a constant for large $|z|$, i.e.

$$f'(z) = 2A - \overline{W'(0)} + \cdots \quad \text{as } z \to \infty. \tag{3.11}$$

3.2. The solution $(u_{\alpha}^{(o)}, \varepsilon_{\alpha\beta}^{(o)}, \tau_{\alpha\beta}^{(o)})$

This elastic field is governed by the boundary conditions Eqs. (2.10) and (2.11) for $R = \infty$. Let $f^{(\mathrm{o})}(z)$ and $W^{(\mathrm{o})}(z)$ denote the solution. The traction-free condition, together with Eqs. (3.4)–(3.11), gives

$$f^{(\mathrm{o})}(z) = (A - \Omega)z - C\frac{\rho^2}{z}, \tag{3.12}$$

$$2W^{(\mathrm{o})}(z) - f^{(\mathrm{o})}(z) = \left(A + \Omega - C\frac{\rho}{z}\right)X(z), \tag{3.13}$$

where

$$X(z) = \left[(z - z_0)(z - \bar{z}_0)\right]^{1/2}, \quad z_0 = \rho e^{\mathrm{i}\phi}, \tag{3.14}$$

$$\Omega = \overline{W^{(\mathrm{o})\prime}(0)} = \left[4A(1 + \cos\phi) + C\sin^2\phi\right]/2(3 - \cos\phi). \tag{3.15}$$

This is the circular-arc crack solution in its most condensed form (cf. Cotterell and Rice, 1980).

The SIFs at $z = z_0 = \rho e^{\mathrm{i}\phi}$, denoted by $K_{\mathrm{I}}^{(\mathrm{o})}$ and $K_{\mathrm{II}}^{(\mathrm{o})}$, are conveniently defined by

$$K_{\mathrm{I}}^{(\mathrm{o})} - \mathrm{i}K_{\mathrm{II}}^{(\mathrm{o})} = (\pi\rho\sin\phi)^{1/2} e^{-\mathrm{i}\phi/2}\left[\frac{\sigma_{11} + \sigma_{22}}{3 - \cos\phi} + \frac{\sigma_{22} - \sigma_{11}}{2}\left(\frac{3 - \cos\phi}{2} - \frac{4}{3 - \cos\phi} + \mathrm{i}\sin\phi\right)\right]. \tag{3.16}$$

Setting $\rho\sin\phi = a$ and $\phi = 0$ in the above, we obtain $K_{\mathrm{I}}^{(\mathrm{o})} - \mathrm{i}K_{\mathrm{II}}^{(\mathrm{o})} = \sigma_{11}\sqrt{\pi a}$, as it should be.

The displacement along the crack periphery is obtained by

$$\left[u_r^{(o)}(\rho,\theta)+\mathrm{i}u_\theta^{(o)}(\rho,\theta)\right]^{\pm} = -\frac{\kappa+1}{2\mu}\left[W^{(o)}-f^{(o)}\right]^{\mp}\mathrm{e}^{-\mathrm{i}\theta}$$
$$= -\frac{\kappa+1}{4\mu}\left\{\left[(A+\Omega)\mathrm{e}^{-\mathrm{i}\theta}-C\mathrm{e}^{-\mathrm{i}2\theta}\right]X^{\mp}\left(\rho\mathrm{e}^{\mathrm{i}\theta}\right)-\left[(A-\Omega)-C\mathrm{e}^{-\mathrm{i}2\theta}\right]\rho\right\}. \quad (3.17)$$

It follows from the above that

$$u_\theta^{(o)}(\rho,\phi)-u_\theta^{(o)}(\rho,-\phi)=\frac{(\kappa+1)\rho}{2\mu}C\sin 2\phi, \quad (3.18)$$

$$u_r^{(o)}(\rho+0,\theta)+u_r^{(o)}(\rho-0,\theta)=\frac{(\kappa+1)\rho}{2\mu}\left[-C\cos 2\theta+\frac{4A(1-\cos\phi)-C\sin^2\phi}{2(3-\cos\phi)}\right]. \quad (3.19)$$

Thus, the elastic interaction energy defined by Eq. (2.22) becomes

$$U_{\mathrm{e}}^{(\mathrm{os})}=-\frac{(\kappa+1)\rho\Sigma_0}{2\mu}\left[C+\frac{4A(1-\cos\phi)-C\sin^2\phi}{(3-\cos\phi)}\phi\right]. \quad (3.20)$$

We proceed to calculate $U_{\mathrm{e}}^{(o)}$. This elastic energy is defined by Eq. (2.18) and may be written as the sum of two terms, viz.

$$U_{\mathrm{e}}^{(o)}=U_{\mathrm{e}0}^{(o)}+\Delta U_{\mathrm{e}}^{(o)}, \quad (3.21)$$

where

$$U_{\mathrm{e}0}^{(o)}=\frac{\pi R^2}{16\mu}\left[(\kappa-1)(\sigma_{11}+\sigma_{22})^2+2(\sigma_{11}-\sigma_{22})^2\right] \quad (3.22)$$

is the energy of the solid without the crack. In order to obtain $\Delta U_{\mathrm{e}}^{(o)}$ by applying Eq. (2.18), $\mathbf{u}^{(o)}$ must be solved for finite R. Our solution (3.12)–(3.19) are for $R=\infty$. However, $\Delta U_{\mathrm{e}}^{(o)}$ may also be obtained from another formula (Sih and Liebowitz, 1968; Wu, 1978a,b). It is

$$\Delta U_{\mathrm{e}}^{(o)}=-\frac{\pi(\kappa+1)}{2\mu}\mathrm{Re}(W_1w_{-1}+W_{-1}w_1), \quad (3.23)$$

where the four coefficients are defined by Eqs. (3.4)–(3.7) and

$$w_{-1}=-\rho^2\left[\frac{4A(1-\cos\phi)-C\sin^2\phi}{(3-\cos\phi)}\right], \quad (3.24)$$

$$W_{-1}=-\rho^2\left[\frac{C}{2}(1-\cos\phi)-A\frac{\sin^2\phi}{3-\cos\phi}-C\frac{\sin^4\phi}{8(3-\cos\phi)}\right]. \quad (3.25)$$

Substituting the above into Eq. (3.23), we obtain

$$\Delta U_{\mathrm{e}}^{(o)}=\frac{\pi(\kappa+1)}{2\mu}\rho^2\left\{\frac{4(1-\cos\phi)}{3-\cos\phi}A^2-\frac{2\sin^2\phi}{3-\cos\phi}AC+\left[\frac{1}{2}(1-\cos\phi)-\frac{\sin^4\phi}{8(3-\cos\phi)}\right]C^2\right\}. \quad (3.26)$$

It can be verified by a direct but tedious differentiation that

$$\frac{1}{2\rho}\frac{\partial\Delta U_{\mathrm{e}}^{(o)}}{\partial\phi}=\frac{\kappa+1}{8\mu}\left[K_{\mathrm{I}}^{(o)^2}+K_{\mathrm{II}}^{(o)^2}\right], \quad (3.27)$$

as it should be.

3.3. The solution $(u_\alpha^{(s)}, \varepsilon_{\alpha\beta}^{(s)}, \tau_{\alpha\beta}^{(s)})$

This elastic field is governed by the boundary conditions (2.12)–(2.14) for $R = \infty$. Let $W^{(s)}(z)$, $w^{(s)}(z)$ and $f^{(s)}(z)$ be the complex functions associated with the desired solution. The elastic field associated with a crack-tip point load is known (England, 1971). For the $2\Sigma_0$ force at $z = z_0$, the three functions must tend to $W_1(z)$, $w_1(z)$ and $f_1(z)$ as $z \to z_0$, where

$$W_1(z) = -\frac{F}{4\pi}\ln(z - z_0), \tag{3.28}$$

$$w_1(z) = \frac{\overline{F}}{4\pi}\ln(z - z_0) + \frac{F}{4\pi}\frac{\bar{z}_0}{z - z_0}, \tag{3.29}$$

and $F = 2\Sigma_0(\sin\phi - \mathrm{i}\cos\phi)$. The functions must also tend to $W_2(z)$, $w_2(z)$ and $f_2(z)$ as $z \to \bar{z}_0$, where

$$W_2(z) = -\frac{\overline{F}}{4\pi}\ln(z - \bar{z}_0), \tag{3.30}$$

$$w_2(z) = \frac{F}{4\pi}\ln(z - \bar{z}_0) + \frac{\overline{F}}{4\pi}\frac{z_0}{z - \bar{z}_0}. \tag{3.31}$$

The associated $f_1(z)$ and $f_2(z)$ may be deduced by applying Eq. (3.3). They are

$$f_1'(z) = \frac{F}{4\pi z_0}\left[1 - \frac{2z_0}{z - z_0} + \frac{z_0}{z}\right], \tag{3.32}$$

$$f_2'(z) = \frac{\overline{F}}{4\pi\bar{z}_0}\left[1 - \frac{2\bar{z}_0}{z - \bar{z}_0} + \frac{\bar{z}_0}{z}\right]. \tag{3.33}$$

The traction conditions (2.13) are satisfied by

$$f^{(s)\prime+} - f^{(s)\prime-} = 2\Sigma_0/\rho, \tag{3.34}$$

$$\left(2W^{(s)\prime} - f^{(s)\prime}\right)^+ + \left(2W^{(s)\prime} - f^{(s)\prime}\right)^- = 0. \tag{3.35}$$

For large $|z|$,

$$W^{(s)\prime}(z) = \mathrm{O}\left(\frac{1}{z^2}\right), \tag{3.36}$$

$$f^{(s)\prime}(z) = -\overline{W^{(s)\prime}(0)}, \tag{3.37}$$

and $f^{(s)\prime}$ has no pole at the origin. It is now convenient to obtain the solution in the following form:

$$W^{(s)} = W_1(z) + W_2(z) + W_0(z), \tag{3.38}$$

$$w^{(s)} = w_1(z) + w_2(z) + w_0(z), \tag{3.39}$$

$$f^{(s)} = f_1(z) + f_2(z) + f_0(z). \tag{3.40}$$

Making all the necessary substitutions, we obtain from Eqs. (3.34) and (3.35)

$$f_0'^+ - f_0'^- = 2\Sigma_0/\rho, \tag{3.41}$$

$$\left(2W_0' - f_0'\right)^+ + \left(2W_0' - f_0'\right)^- = \frac{2\Sigma_0 \sin\phi}{\pi}\frac{1}{\zeta}, \tag{3.42}$$

where $\zeta = \rho e^{i\theta}$. It may also be deduced from Eqs. (3.36) and (3.37) that

$$W_0'(z) = \frac{\Sigma_0 \sin\phi}{\pi}\frac{1}{z} + \cdots, \tag{3.43}$$

$$f_0'(z) = \frac{\Sigma_0 \sin\phi}{\pi} - \overline{W_0'(0)} + \cdots, \tag{3.44}$$

for large $|z|$. The requirement that $f^{(s)\prime}(z)$ has no pole at $z = 0$ leads to the condition that $f_0'(z)$ actually has a simple pole, viz.

$$f_0'(z) = -\frac{\Sigma_0 \sin\phi}{\pi}\frac{1}{z} \quad \text{as } z \to 0. \tag{3.45}$$

It follows from Eqs. (3.41)–(3.45) that

$$f_0'(z) = \frac{\Sigma_0}{i\pi\rho}\ln\frac{z - \bar{z}_0}{z - z_0} - \frac{\Sigma_0 \sin\phi}{\pi}\frac{1}{z} - \Omega_0, \tag{3.46}$$

$$2W_0'(z) - f_0'(z) = \frac{\Sigma_0 \sin\phi}{\pi}\frac{1}{z} + \Omega_0 X'(z), \tag{3.47}$$

where

$$\Omega_0 = \overline{W_0'(0)} = -\frac{2\Sigma_0\phi}{\pi\rho(3 - \cos\phi)}. \tag{3.48}$$

The SIFs are determined from the function $X'(z)$ of Eq. (3.47). Let $K_{\mathrm{I}}^{(s)}$ and $K_{\mathrm{II}}^{(s)}$ denote the SIFs at $z = z_0$. We have

$$K_{\mathrm{I}}^{(s)} - iK_{\mathrm{II}}^{(s)} = -\frac{\Sigma_0\phi}{1 + \sin^2\frac{\phi}{2}}\left(\frac{\sin\phi}{\pi\rho}\right)^{1/2}\left(\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\right). \tag{3.49}$$

Finally, integrating Eqs. (3.46) and (3.47) and using Eqs. (3.38)–(3.40), we get

$$f(z) = \frac{i\Sigma_0}{\pi\rho}zY(z) - \frac{2\Sigma_0}{\pi}\sin\phi - \Omega_0 z, \tag{3.50}$$

$$W(z) = \frac{i\Sigma_0}{2\pi\rho}zY(z) - \frac{\Sigma_0}{\pi}\sin\phi - \frac{1}{2}\Omega_0[z - X(z)], \tag{3.51}$$

where

$$Y(z) = \ln\frac{z - z_0}{z - \bar{z}_0}. \tag{3.52}$$

The displacement along the crack periphery may now be calculated from Eqs. (3.1), (3.50)–(3.52). It is

$$(u_r + \mathrm{i}u_\theta)^\pm = \frac{1}{2\mu}\mathrm{e}^{-\mathrm{i}\theta}\left(\kappa W^\pm - W^\mp + f^\mp\right)$$
$$= \pm\frac{(\kappa-1)\Sigma_0}{4\mu} + \frac{(\kappa+1)\Sigma_0}{4\pi\mu}\left\{\left[-2\sin\phi\cos\theta - \frac{\phi(1-\cos\phi)}{(3-\cos\phi)}\right] + \mathrm{i}[2\sin\phi\sin\theta + \ln\delta] - \frac{2\phi}{\rho(3-\cos\phi)}X^\pm\mathrm{e}^{-\mathrm{i}\theta}\right\} \tag{3.53}$$

where $\delta = |z - z_0|/|z - \bar{z}_0|$ and the property of $Y^\pm$ has been used. It is also noted that $X^- = -X^+$, so that they do not contribute to the value of $U_\mathrm{e}^{(\mathrm{s})}$, Eq. (2.19). Applying the previous result, we obtain from Eq. (2.19)

$$U_\mathrm{e}^{(\mathrm{s})} = -\frac{(\kappa+1)\Sigma_0^2}{2\pi\mu}\ln\delta_0 + \Delta U_\mathrm{e}^{(\mathrm{s})}, \tag{3.54}$$

$$\Delta U_\mathrm{e}^{(\mathrm{s})} = \frac{(\kappa+1)\Sigma_0^2}{2\pi\mu}\frac{(1-\cos\phi)}{(3-\cos\phi)}\phi^2, \tag{3.55}$$

where $\ln\delta_0$ indicates the logarithmic singularity when z tends to z_0 or $\bar{z}_0$.

It is now possible to compare Eq. (3.55) with Eq. (3.26) to obtain

$$\Delta U_\mathrm{e}^{(\mathrm{s})}/\Delta U_\mathrm{e}^{(\mathrm{o})} \propto (\Sigma_0/\sigma_\mathrm{A}\rho)^2, \tag{3.56}$$

where σ_A indicates the magnitude of an applied stress. Since $\sigma_\mathrm{A}^2\pi\rho\sin\phi \propto K_\mathrm{IC}^2 = 16\mu\Gamma_0/(\kappa+1)$, the significance of $\Delta U_\mathrm{e}^{(\mathrm{s})}$, relative to that of $\Delta U_\mathrm{e}^{(\mathrm{o})}$, is controlled by the factor defined by

$$\frac{\Delta U_\mathrm{e}^{(\mathrm{s})}}{\Delta U_\mathrm{e}^{(\mathrm{o})}} \propto \left(\frac{\Sigma_0}{\sigma_\mathrm{A}\rho}\right)^2 \propto \frac{\pi(\kappa+1)}{16}\sin\phi\left(\frac{\Sigma_0}{\Gamma_0}\right)\left(\frac{\Sigma_0}{\rho\mu}\right), \tag{3.57}$$

which indicates that ρ has to be small in order for the last ratio to be of significance. Differentiating Eq. (3.55), we obtain

$$\frac{1}{2\rho}\frac{\partial\Delta U_\mathrm{e}^{(\mathrm{s})}}{\partial\phi} = \frac{\kappa+1}{8\mu}\left[K_\mathrm{I}^{(\mathrm{s})^2} + K_\mathrm{II}^{(\mathrm{s})^2}\right] + \frac{(\kappa+1)\Sigma_0^2\phi\sin^2\frac{\phi}{2}}{2\pi\mu\rho\left(1+\sin^2\frac{\phi}{2}\right)}, \tag{3.58}$$

which indicates that the energy release rate is not proportional to the sum of squares of $K_\mathrm{I}^{(\mathrm{s})}$ and $K_\mathrm{II}^{(\mathrm{s})}$, a consequence of the point load at the crack tips, Eq. (2.8).

3.4. The potential energy of the bulk-surface system

This is the function Π given by Eq. (2.27). With respect to Eqs. (2.20), (3.21) and (3.54), Π may be written as

$$\Pi = \Pi_0 + \Delta\Pi, \tag{3.59}$$

$$\Delta\Pi = -\Delta U_\mathrm{e}^{(\mathrm{o})} - \Delta U_\mathrm{e}^{(\mathrm{s})} - \Delta U_\mathrm{e}^{(\mathrm{os})} + 4\rho\phi\Gamma_0, \tag{3.60}$$

where Π_0 is independent of the crack-configuration parameters ρ and ϕ. The function $\Delta\Pi$ owes its existence to ρ and ϕ. It may therefore be referred to as a *configurational energy*. The configurational equilibrium of a system is determined by the stationary of its configurational energy. We examine the stationary characteristics of Eq. (3.60) for the case $\sigma_{11} = \sigma_{22} = \sigma$.

The stress intensity factors for $\sigma_{11} = \sigma_{22} = \sigma$ are first calculated from Eqs. (3.16) and (3.49). They are

$$K_{\mathrm{I}} = K_{\mathrm{I}}^{(\mathrm{o})} + K_{\mathrm{I}}^{(\mathrm{s})} = \left(\sigma - \frac{\Sigma_0 \phi}{\pi\rho}\right) \frac{\sqrt{\pi\rho \sin\phi}}{1 + \sin^2\frac{\phi}{2}} \cos\frac{\phi}{2}, \tag{3.61}$$

$$K_{\mathrm{II}} = K_{\mathrm{II}}^{(\mathrm{o})} + K_{\mathrm{II}}^{(\mathrm{s})} = \left(\sigma - \frac{\Sigma_0 \phi}{\pi\rho}\right) \frac{\sqrt{\pi\rho \sin\phi}}{1 + \sin^2\frac{\phi}{2}} \sin\frac{\phi}{2}, \tag{3.62}$$

where Σ_0 signifies the surface-stress contribution. For convenience, we also define K and $K^{(\mathrm{o})}$ by

$$K^2 = K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2 = \left(\sigma - \frac{\Sigma_0 \phi}{\pi\rho}\right)^2 \pi\rho \sin\phi \bigg/ \left(1 + \sin^2\frac{\phi}{2}\right)^2, \tag{3.63}$$

$$K^{(\mathrm{o})^2} = K_{\mathrm{I}}^{(\mathrm{o})^2} + K_{\mathrm{II}}^{(\mathrm{o})^2} = \sigma^2 \pi\rho \sin\phi \bigg/ \left(1 + \sin^2\frac{\phi}{2}\right)^2. \tag{3.64}$$

The associated configurational energy is now obtained by substituting Eqs. (2.20), (3.26) and (3.55) into Eq. (3.59), viz.

$$\Delta\Pi = 4\rho\phi\Gamma_0 - \frac{(\kappa+1)}{2\mu} \frac{(1-\cos\phi)}{(3-\cos\phi)} \left(\sqrt{\pi}\sigma\rho - \frac{\Sigma_0\phi}{\sqrt{\pi}}\right)^2. \tag{3.65}$$

By setting to zero the derivative $\partial\Delta\Pi/2\rho\partial\phi$, we obtain the configurational equilibrium condition for the circularly extending circular-arc crack. The condition is

$$K^2 - \frac{4\Sigma_0 \sin^2\frac{\phi}{2}}{\sqrt{\pi\rho\sin\phi}} K - K_{\mathrm{IC}}^2 = 0, \tag{3.66}$$

where K_{IC} is the constant defined by Eq. (1.8). It follows from Eqs. (3.63), (3.64) and (3.66) that

$$K^{(\mathrm{o})^2} \geqslant K^2 \geqslant K_{\mathrm{IC}}^2, \tag{3.67}$$

where equalities hold only if $\Sigma_0 = 0$. Thus, the inclusion of surface stress is to increase the apparent toughness of the material. In terms of the applied σ, Eq. (3.66) may be solved to yield

$$\sigma\sqrt{\pi\rho\sin\phi} \approx \left(1 + \sin^2\frac{\phi}{2}\right) K_{\mathrm{IC}} + \frac{2\Sigma_0 \sin^2\frac{\phi}{2}(1 + \sin^2\frac{\phi}{2})}{\sqrt{\pi\rho\sin\phi}} + \frac{\Sigma_0\phi\sin\phi}{\sqrt{\pi\rho\sin\phi}} + \cdots, \tag{3.68}$$

which may be compared with Eq. (1.11).

4. Summary

The fully nonlinear surface chemical potential for elastic solids undergoing surface accretion – as delineated by Leo and Sekerka (1989), Wu (1996), Freund (1998) and Norris (1998) – is now well established. The coupling between the 'surface' and the bulk material is, in general, highly nonlinear, and experimental values for solid surface stress are mostly unavailable. However, there is a special case of the general chemical potential, which makes use of the simplicity of the Laplace–Herring model of Grinfeld (1994). It corresponds to the linear surface energy density of Eq. (1.9). This choice, together with the use of linear elasticity for the bulk material, enables us to remove the coupling from the nonlinear problem in that the inclusion of a surface stress is but the addition of another load, which may be separately considered for its contribution to the total deformation. The desired coupling is only entwined in the energy – as is typical of all linear problems – and the result is a modified configurational equilibrium condition. Also, the single constant surface-stress coefficient can be easily carried along the analysis as a parameter. The goal of this work and that of Wu and Wang (2000) is to take advantage of this described simplification to examine the implication of surface stress on stress intensity factors and, above all, on configurational equilibrium conditions.

It is shown that for linear cracks the inclusion of surface stress does not change the values of the stress intensity factors, although the stress field around a crack tip actually becomes more singular than an inverse square root singularity. There is no inconsistency in this result, as the configurational equilibrium is always an energy condition and never a stress criterion. Indeed, the calculated configurational equilibrium is affected by the presence of surface stress, and the explicit result conveyed by Eqs. (1.10) and (1.11) attest to this conclusion.

For curvilinear cracks, the presumably traction-free crack surfaces are now loaded by a uniformly distributed compression on the convex side and a uniformly distributed tension on the concave side. This additional load alters the values of the stress intensity factors, as well as the configurational equilibrium conditions. Eqs. (3.60) and (3.61) indicate the change in stress intensity factors, and Eq. (3.68) expresses the explicit change in terms of energy.

In general, numerical or finite-element based techniques may be applied to compute the modified stress intensity factors. However, if the configurational equilibrium condition – i.e. the fracture criterion – is not a known function of the stress intensity factors, there is really no apparent need to calculate them. As to the numerical determination of the problem-specific configurational equilibrium condition, it is still a very much open question.

It all comes down to the question of whether the inclusion of surface stress is to affect the solution in any significant way. Without the availability of a truly nonlinear solution, the best estimation we can have is (3.57), which indicates a second order effect strongly influenced by the value of the surface stress coefficient.

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